بسم الله الرحمن الرحيم

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جامعة أم القرى
كلية العلوم التطبيقية

نموذج رقم (٨)

إجازة أطروحة علمية في صيغتها النهائية بعد إجراء التعديلات

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المشرف	مناقش من داخل القسم	مناقش من خارج القسم
الاسم : د.محمد رمضان شاهين	الاسم : د. عبدالله عبدالرحمن عبده	الاسم : محمد حسين فهمي
التوقيع :	التوقيع :	التوقيع :

يعتمد :
رئيس قسم العلوم الرياضية

د. عبدالفتاح بن قاري بخاري

المملكة العربية السعودية
وزارة التعليم العالي
جامعة أم القرى
كلية العلوم التطبيقية
قسم العلوم الرياضية

حول خاصية برج سيلو المعممة للزمر المنتهية

إعداد الطالبة

وفاء محمد عبدالقادر فقيه

إشراف

د. محمد رمضان شاهين

الأستاذ المساعد بقسم العلوم الرياضية

كلية العلوم التطبيقية — جامعة أم القرى

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إهـــــــداء

إلى والدي الحبيب ... أهدي ثمرةً زرع بذورها بيديه ثم رعاهـــا وتعـــهدها بتوجيهه وتشجيعه حتى آتت أكلها .

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ملخص البحث

تهتم الرسالة بإعطاء تعميم لخاصية برج سيلو لزمرة منتهية ، ومن ثم دراسة بعض الخصائص لخاصية برج سيلو المعممة لتلك الزمرة ، بالإضافة إلى دراسة تأثير بعض الزمر الجزئية التي تملك خاصية برج سيلو المعممة على بنية الزمرة المنتهية .

الزمرة الجزئية H من الزمرة المنتهية G تكون (p, q)-زمرة جزئية إذا حققت الشروط الآتية :

(١) رتبة H تحتوي على العاملين الأوليين المختلفين p ، q فقط .

(٢) H ليست متلاشية (non-nilpotent) بينما جميع الزمر الجزئية الفعلية منها تكون متلاشية.

(٣) الزمرة الجزئية المبدلة (Commutator) H' من H تكون زمرة p-سيلو (Sylow) الجزئية من H .

وبهذا فإن التعبير $G \leq (p, q)$ يعني أن G تحتوي على (p, q)-زمرة جزئية خلاف ذلك فإن $G \not\leq (p, q)$ يعني أن G لا تحتوي على (p, q)-زمرة جزئية .

من المعلوم أنه إذا كانت G زمرة رتبتها $p_1^{e_1} \; p_2^{e_2} \; \dots \; p_n^{e_n}$ حيث أن $p_1 > p_2 > \dots > p_n$ ولتكن P_i هي زمرة p_i-سيلو الجزئية من G فإننا نقول أن G تملك خاصية سيلو المرتبة (Ordered Sylow tower property) إذا كانت $P_1 \; P_2 \; \dots \; P_k$ زمرة جزئية ناظمية (normal) من الزمرة G لكل $k = 1, 2, \dots, n$. ففي عام ١٩٦٩م أثبت كورادي (Corradi) في مقالته [4] أنه إذا كانت G زمرة رتبتها $p_1^{e_1} \; p_2^{e_2} \; \dots \; p_n^{e_n}$ حيث $p_1 > p_2 > \dots > p_n$ فإن G تملك خاصية برج سيلو المرتبة إذا وإذا فقط $G \not\leq (p_k \; , \; p_m)$ لكل $k \, , \, m \leq n$ حيث $k > m$. وفي هذه الرسالة نعطي التعريف الآتي الذي يُعد تعميماً لخاصية برج سيلو المرتبة :

إذا كانت G زمرة رتبتها $p_1^{e_1}\ p_2^{e_2}\ \dots\ p_n^{e_n}$ بحيث أن $p_1 > p_2 > \dots > p_n$ فإن G تملك خاصية برج سيلو المعممة (generalized Sylow tower property) إذا وجدت تبديلة σ في S_n (زمرة التباديل على n حرف) بحيث أن $P_{\sigma(1)}\ P_{\sigma(2)}\ \dots\ P_{\sigma(k)}$ زمرة جزئية ناظمية من G لكل n , ... , $k = 1, 2$ ، حيث $P_{\sigma(i)}$ زمرة $p_{\sigma(i)}$-سيلو الجزئية من G. فباستخدام هذا التعريف نلاحظ أن :

(١) خاصية برج سيلو المرتبة لزمرة ما فإنها تكون برج سيلو المعممة لتلك الزمرة ، العكس غير صحيح ويتضح ذلك بفحص A_4 ، حيث A_4 هي زمرة جميع التباديل الزوجية على أربع حروف .

(٢) إذا كانت G تملك خاصية برج سيلو المعممة فإن زمرة $p_{\sigma(1)}$-سيلو الجزئية تكون ناظمية في G وأن G لها $p_{\sigma(n)}$ - مكملاً ناظمياً (normal $p_{\sigma(n)}$-complement) .

(٣) لأي زمرة متلاشية من الرتبة $p_1^{e_1}\ p_2^{e_2}\ \dots\ p_n^{e_n}$ حيث $p_1 > p_2 > \dots > p_n$ فإنها تملك خاصية برج سيلو المعممة لأي σ في S_n .

(٤) أي زمرة فوق قابلة للحل (supersolvable) من الرتبة $p_1^{e_1}\ p_2^{e_2}\ \dots\ p_n^{e_n}$ حيث $p_1 > p_2 > \dots > p_n$ تملك خاصية برج سيلو المعممة إذا اعتبرنا أن σ هي تبديلة الوحدة في S_n (أنظر [١٢ ، ص٦]) والعكس غير صحيح حيث أنه إذا كانت $G = A_4$ فإن G زمرة لها خاصية برج سيلو المعممة ولكنها ليست فوق قابلة للحل .

(٥) الزمر المتلاشية $\subset$ الزمر فوق القابلة للحل $\subset$ الزمر التي تملك خاصية برج

سيلو المرتبة ⊃ الزمرة التي تملك خاصية برج سيلو المعممة ⊃ الزمر القابلة للحل (solvable groups) .

تتضمن هذه الرسالة ثلاثة أبواب :

الباب الأول : يتضمن المقدمة والمصطلحات المختلفة والتعاريف الأساسية المستخدمة خلال هذه الرسالة .

الباب الثاني : يتضمن عدداً من النتائج المعروفة في المراجع ، لأنها تلعب دوراً هاماً في إثبات النتائج التي حصلنا عليها خلال هذه الرسالة . بالإضافة إلى ذلك يتضمن هذا الباب بعض النتائج المتعلقة بخاصية برج سيلو المرتبة للزمرة المنتهية .

الباب الثالث : يختص هذا الباب بالنتائج الجديدة التي حصلنا عليها عن خاصية برج سيلو المعممة لزمرة منتهية والتي تعتبر تعميماً لبعض النتائج المعروفة سابقاً . وأكثر تحديداً سنثبت النتائج التالية :

(١) بفرض أن G زمرة رتبتها $p_1^{e_1}\ p_2^{e_2}\ \ldots\ p_n^{e_n}$ بحيث $p_1 > p_2 > \ldots > p_n$ فإن G تملك خاصية برج سيلو المعممة إذا وإذا فقط إذا وجدت تبديلة σ في S_n بحيث أن $G \not\leq (p_{\sigma(i)}, p_{\sigma(j)})$ لكل زوج من الأعداد الأولية $p_{\sigma(i)}$ و $p_{\sigma(j)}$ ينتمي إلى $\pi(G)$ بحيث أن $i > j$ و $1 \leq i\ ,\ j \leq n$.

(٢) بفرض أن G زمرة قابلة للحل ذات الرتبة $p_1^{e_1}\ p_2^{e_2}\ \ldots\ p_n^{e_n}$ حيث أن $p_1 > p_2 > \ldots > p_n$ ولتكن σ تبديلة ما في S_n ، وبفرض أن H هي زمرة $p_{\sigma(1)}$ الجزئية الناظمية من G بحيث أن G/H تملك خاصية برج سيلو المعممة فإن G تملك هذه الخاصية لنفس التبديلة .

(٣) وبفرض أن G زمرة قابلة للحل وأن H تكون π-Hall زمرة جزئية من G بحيث أن $H \leq \Phi(G)$ حيث $\Phi(G)$ هي زمـــرة الفراتيـــني (Frattini) الجزئية من G . فإن H تملك خاصية برج سيلو المعممة إذا وإذا فقط كانت $H/\Phi(G)$ تملك خاصية برج سيلو المعممة .

(٤) إذا كـــانت G زمـــرة رتبتـــها $p_1^{e_1}\ p_2^{e_2}\ \dots\ p_n^{e_n}$ حيـــــث أن $p_1 > p_2 > \dots > p_n$ وبفرض أن G تملـــك ثـــلاث زمـــر جزئيـــة H_1, H_2, H_3 بحيث أن $\left(|G:H_i|, |G:H_j|\right) = 1$ لكـــل $1 \leq i \neq j \leq 3$. فإنه إذا كانت H_i حيث $1 \leq i \leq 3$ ، تملك خاصية برج ســـيلو المعممـــة لتبديلة ما σ من S_n فإن G تملك خاصية برج سيلو المعممـــة لنفـــس التبديلة .

(٥) بفرض أن G زمرة منتهية ، فإن G لا تملك خاصية برج سيلو المعممة بينمـا جميع زمرها الجزئية الفعلية تملك تلك الخاصية إذا وإذا فقط كانت G ليســت متلاشية وجميع زمرها الجزئية الفعلية متلاشية .

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NOTE

Besides the work carried out in the thesis, the student attended and passed on successfully the following postgraduate courses:

(1) Functional Analysis.

(2) Algebra.

(3) Representation theory of finite groups.

(4) Ring theory.

(5) Group theory.

(6) General Topology.

Head of the Department of Mathematical Science

Dr. Abd Al-Fattah Gari

ABSTRACT

The object of this thesis is to generalize the concept of the ordered Sylow tower property of a finite group and we investigate the influence of the generalized Sylow tower property of some subgroups of a group on its structure.

CONTENTS

CHAPTER I

GENERAL INTRODUCTION

GENERAL INTRODUCTION

1.1 INTRODUCTION

Throughout this thesis the term " group " always means a group of finite order. *A* subgroup H of a group G is called a (p, q)-subgroup if: **(i)** the order of H involves only prime factors p and q ; **(ii)** H is not nilpotent and all its proper subgroups are nilpotent ; **(iii)** the derived group H' is the Sylow p-subgroup of H. We write $(p, q) \leq G$ if G contains a (p, q)-subgroup, otherwise, we shall write $(p, q) \nleq G$ (see [**8** , p.291]) . It is known that : If G is a group of order $p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$, where $p_1 > p_2 > \ldots > p_n$ and P_i be a Sylow p_i-subgroup of G, then we say that G possesses an ordered Sylow tower property if $P_k \ldots P_2 P_1$ is a normal subgroup of G for $k = 1, 2, \ldots, n$. In [**4**], Corrádi proved that : Let G be a group of order $p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$, where $p_1 > p_2 > \ldots > p_n$. Then G possesses an ordered Sylow tower property if and only if $(p_k , p_m) \nleq G$ for $k , m \leq n$ where $k > m$. In this thesis we introduce the following definition which may be considered as a generalization of the concept of the ordered Sylow tower property. Let G be a group of order $p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$, where $p_1 > p_2 > \ldots > p_n$. We say that G possesses a generalized Sylow tower property if for some σ in S_n, $P_{\sigma(k)} \ldots P_{\sigma(2)} P_{\sigma(1)}$ is a normal subgroup of G for $k = 1, 2, \ldots, n$, where $P_{\sigma(i)}$ is a Sylow $p_{\sigma(i)}$-subgroup of G. Using this definition we note that :

(1) Every ordered Sylow tower property is a generalized Sylow tower property. The converse is not true. This can be easily shown by examining A_4, the alternating group of degree four.

(2) If G possesses a generalized Sylow tower property, then G has a normal Sylow $p_{\sigma(1)}$-subgroup and G has a normal $p_{\sigma(n)}$-complement.

(3) Any nilpotent group of order $p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$, has a generalized Sylow tower property for any σ is S_n.

(4) Any supersolvable group of order $p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$, has a generalized Sylow tower property if σ is the identity permutation in S_n (see [**11**, p.6]). The converse is not true. Set $G = A_4$. Clearly, G has a generalized Sylow tower property but G is not supersolvable.

(5) The relationships among several important classes of solvable groups may be summarized in the following form :
Nilpotent groups $\subset$ supersolvable groups $\subset$ groups which have the ordered Sylow tower property $\subset$ groups which have the generalized Sylow tower property $\subset$ solvable groups.

The object of this thesis is to investigate the conditions under which a group possesses the generalized Sylow tower property and we also obtain some properties concerning the generalized Sylow tower property. Furthermore, we study the influence of the generalized Sylow tower property of some subgroups of a group on its structure.

This thesis includes three chapters :

CHAPTER I. This chapter is an explanatory which contains an introduction, a various notations and basic definitions that will be used throughout this thesis.

CHAPTER II. In this chapter, we list a number of well known results which will be used throughout this thesis, referring the reader to their proofs in the literature. We also present some results on the concept of the ordered Sylow tower property of a group.

CHAPTER III. This chapter contains several new results on the generalized Sylow tower property of a group and some results concerning the properties of the generalized Sylow tower property are obtained. Furthermore, we investigate the influence of the generalized Sylow tower property of a group on its structure. More precisely, we prove the following results:

(1) Let G be a group of order $p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$. Then G possesses a generalized Sylow tower property if and only if for some σ in S_n, $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(G)$ with $i > j$, $1 \leq i, j \leq n$.

(2) Let G be a solvable group of order $p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$. For some σ in S_n, let H be a normal $p_{\sigma(1)}$-subgroup of G such that G/H possesses a generalized Sylow tower property. Then G so also does.

(3) Let G be a solvable group. Suppose that H is a π-Hall subgroup of G such that $\Phi(G) \leq H$. Then H possesses a generalized Sylow tower property if and only if $H/\Phi(G)$ possesses a generalized Sylow tower property.

(4) Let G be a group of order $p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$. Suppose that G has three subgroups H_1, H_2 and H_3 whose indices in G are pairwise relatively prime. If H_1, H_2 and H_3 possesses the generalized Sylow tower property for some σ in S_n, then G possesses a generalized Sylow tower property for the same σ.

(5) Let G be a group. Then G is a minimal non-generalized Sylow tower if and only if G is a minimal non-nilpotent group.

1.2 NOTATION , DEFINITIONS AND BASIC PROPERTIES

Groups and subgroups will be denoted by capital letters and group elements will be denoted by small letters. Small letters will also denote positive integers.

We use the symbol " 1 " for both the identity element and the identity subgroup of a group.

The following notations concerning group G used frequently :

$x \in G$	x belongs to G .
$\{ . , . , ... \}$	the set whose members are ...
$\langle ... \rangle$	the subgroup generated by ...
$\mid G \mid$	the order of G .
$p \mid \mid G \mid$	$\mid G \mid$ is divisible by the prime p .
$p \mid \mid G \mid$	$\mid G \mid$ is not divisible by the prime p .
π	set of primes .
$\pi(G)$	the set of distinct primes dividing $\mid G \mid$.
$\mid \pi(G) \mid$	the number of primes in $\pi(G)$.
$H \leq G$	H is a subgroup of G .
$H < G$	H is a proper subgroup of G .
$H \nleq G$	H is not a subgroup of G .
$H \lhd G$	H is a normal subgroup of G .
G/H	the factor group of G by H (H is normal in G).
$\Phi(G)$	the Frattini subgroup of G, that is, the intersection of all maximal subgroups of G .
Aut (G)	the automorphism group of G .

If $H \leq G$ and $K \leq G$, then :

$C_G(H)$	the centeralizer of H in G.
$\mid G : H \mid$	the index of H in G.
$H \cap K$	H intersection K.
$H \cong K$	H is isomorphic to K.
$H \precsim K$	H is isomorphic to a subgroup of K.
$H \times K$	the direct product of H and K.
$(\mid H \mid, \mid K \mid) = 1$	$\mid H \mid$ and $\mid K \mid$ are relatively prime.
H^x	$= x^{-1} H x$
H^G	the normal closure of H in G.
G'	the derived subgroup of a group G.
S_n	the symmetric group of degree n.
A_n	the alternating group of degree n.
$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & & \sigma(n) \end{pmatrix}$	permutation of set $\{1, 2, \ldots, n\}$.

A group G is called a π-group if the order of G is divisible only by primes in π. A π-Hall subgroup of G is a π-subgroup of G whose index in G is not divisible by any prime in π. If $\pi = \{p\}$, then a π-Hall subgroup is a Sylow p-subgroup. The complementary set of primes to π will be denoted by π'. Thus we have also the notation π' and p'-group. For example, a $2'$-group is simply a group of odd order.

If H is a subgroup of G which is invariant under every automorphism of G, then we say that H is characteristic in G and write H char G (hence if H char K and K is normal in G, we have H is normal in G).

Let P be a Sylow p-subgroup of G. We say that G has a normal p-complement if there exists a normal subgroup K of G such that $G = PK$ and $P \cap K = 1$.

Let H be a subgroup of G. The normal closure of H in G is the intersection of all normal subgroups of G which contain H, and is denoted by H^G.

A normal series of G is a finite sequence (A_0 , A_1 , ... , A_n) of subgroups such that $1 = A_0 \lhd A_1 \lhd \ldots \lhd A_n = G$. An invariant series is a normal series such that $A_i \lhd G$, $i=0,1,\ldots n$. The factors of a normal series are the factor groups A_{i+1}/A_i , $0 \leq i \leq n-1$.

A group is solvable if and only if it has a normal series whose factors are abelian . If H is normal in G and both H and G/H are solvable , then G is solvable [**6** ; Theorem 4.1] .

A cyclic invariant series of a group G is an invariant series of G whose factors are cyclic. A group G is supersolvable if and only if it has a cyclic invariant series. If G/H is supersolvable and H is cyclic , then G is supersolvable [**10** ; Theorem 7.2.14] .

A group G is nilpotent if and only if it possesses an invariant series in which each factor A_{i+1}/A_i is in the center of G/A_i for $i = 0, 1, 2, \ldots , n-1$.

A nonempty class $\mathfrak{F}$ of a solvable groups is a formation if: **(1)** If $G \in \mathfrak{F}$ and N is normal in G , then $G/N \in \mathfrak{F}$; **(2)** If N_1 , N_2 are normal subgroups of G such that G/N_1 , $G/N_2 \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$. A formation $\mathfrak{F}$ is saturated if $G/\Phi(G) \in \mathfrak{F}$, then $G \in \mathfrak{F}$. For example : the class of nilpotent groups $\mathfrak{N}$ is a formation [**6** ; Theorem 3.3] . Moreover , $\mathfrak{N}$ is saturated as a group G is nilpotent if and only if $G/\Phi(G)$ is nilpotent [**10** ; Theorem 7.4.8] ; also the class of supersolvable groups $\delta\delta$ is a formation [**11** ; Theorem 1.1] .

Since G is supersolvable if and only if $G/\Phi(G)$ is supersolvable [**11** ; Corollary 3.2] , we have that δδ is saturated.

A subgroup H of G is called a (p, q)-subgroup if : **(i)** the order of H involves only the prime factors p and q ; **(ii)** H is not nilpotent and all its proper subgroups are nilpotent ; **(iii)** the derived subgroup H' is the Sylow p-subgroup of H . We write $(p , q) \leq G$ if G contains a (p, q)-subgroup , otherwise , we shall write $(p , q) \nleq G$ if G contains no (p, q)-subgroup. We refer the reader to [**8** ; p.291] for the standard properties of a (p, q)-subgroup.

CHAPTER II

ON THE ORDERED SYLOW TOWER PROPERTY OF A FINITE GROUP

ON THE ORDERED SYLOW TOWER PROPERTY OF A FINITE GROUP

2.1 INTRODUCTION

In this chapter we collect together the known results that are needed in this thesis, referring the reader to their proofs in the literature. Also , we present some properties of the concept of the ordered Sylow tower property of a finite group.

2.2 KNOWN RESULTS AND THEIR CONSEQUENCES

(**2.2.1**) [**9** ; Theorem 7.47]. Let K be a normal subgroup of G. If K and G/K are solvable , then G is solvable.

(**2.2.2**) [**9** ; Lemma 4.36] . If H is a normal subgroup of G, then $C_G(H)$ is normal in G and $G/C_G(H) \cong$ Aut (H) .

(**2.2.3**) [**6** ; Theorem 3.10] . If G is a cyclic group of prime order p , then Aut (G) is a cyclic group of order $p-1$.

(**2.2.4**) [**10** ; Theorem 13.2.5] . If $G = HK$, $p \mid |G|$ and p is a prime , then there exist a Sylow p-subgroup P_1 of H and a Sylow p-subgroup P_2 of K such that $P_1 P_2$ is a Sylow p-subgroup of G.

(**2.2.5**) [**10**; Exercise 1.6.15]. If G is a group, $H \leq L \leq G$, $K \leq G$ and $G = HK$, then $L = H(L \cap K)$.

As an immediate consequence of Asaad and Shaalan result [**3** ; Corollary 3.5], we have :

(2.2.6) Suppose that H and K are normal subgroups of G of coprime indices and that H and K possess the ordered Sylow tower property. Then G possesses an ordered Sylow tower property.

(2.2.7) [**5** ; Lemma 2 , p. 203], see also [**1** ; Lemma 1 , p.158]. Let p be a fixed prime dividing $|G|$. Then G has a normal p-complement if and only if $(p\ ,\ q) \not\leq G$ for any $q \in \pi(G)$ with $q \neq p$.

(2.2.8) [**5** ; Theorem A]. Let G be a group and $q \neq p$ primes. If $(p\ ,\ q) \not\leq G$, then this property is inherited to every homomorphic image of G.

(2.2.9) [**9** ; Exercise 109] . Let H and K are be normal subgroups of a group G . Then $H \cap K$ is normal in G and $G/H \cap K \cong G/H \times G/K$.

(2.2.10) [**10** ; Exercise 9.3.19]. $\pi(G) = \pi(G/\Phi(G))$.

(2.2.11) [**1** ; Theorem 1]. Let G be a solvable group. Then $(p\ ,\ q) \not\leq G$ if and only if $(p\ ,\ q) \not\leq G/\Phi(G)$, where p and q are fixed distinct primes.

(2.2.12) [**6** ; Theorem 4.1]. Let G be a solvable group and π be a set of primes. Then :

(i) G possesses a π-Hall subgroup.

(ii) Any two π-Hall subgroups of G are conjugate.

(iii) Any π- subgroup of G is contained in a π-Hall subgroup of G.

(2.2.13) [**9** ; Lemma 11.25] . Suppose that G has three solvable subgroups H_1 , H_2 and H_3 whose indices in G are pairwise relatively prime. Then G is solvable.

(2.2.14) [**6** ; Theorem 1.5] . If M is a maximal subgroup of a solvable group , then $|G : M| = p^n$ for some prime p .

(2.2.15) [**6** ; Lemma 4.2] . If H and K are subgroups of G such that $|G : H|$ and $|G : K|$ are relatively primes , then :

(i) $|G : H \cap K| = |G : H| \, |G : K|$.

(ii) $G = HK$.

(2.2.16) [**6** ; Theorem 4.5] . G is a solvable group if and only if G possesses a p'-Hall subgroup for every prime p in $\pi(G)$.

(2.2.17) [**8** ; Satz 5.2] . Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then there exist distinct primes p and q such that :

(i) $|G| = p^m q^n$, where m and n are non-zero natural numbers.

(ii) G has a normal Sylow p-subgroup P and $G/P \cong Q$, where Q is a non-normal cyclic Sylow q-subgroup of G.

2.3 ON THE ORDERED SYLOW TOWER PROPERTY OF A FINITE GROUP

In this section, we prove that :

Theorem 2.3.1 . [**11** ; p.p192-193]. Let G be a group of order $p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$. Suppose that G possesses an ordered Sylow tower property. Then :

(i) Every subgroup of G possesses an ordered Sylow tower property.

(ii) Every homomorphic image of G possesses an ordered Sylow tower property.

Proof . Since G possesses an ordered Sylow tower property, it follows that $P_k \dots P_2 P_1$ is a normal Hall subgroup of G for $k = 1, 2, \dots, n$, where P_i is a Sylow p_i-subgroup in G.

(i) Let H be a subgroup of G. Since $P_k \dots P_2 P_1$ is a normal Hall subgroup of G for $k = 1, 2, \dots, n$, it follows that $(P_k \dots P_2 P_1) \cap H$ is a normal Hall subgroup of H. Thus $P_1 \cap H$ is a normal Sylow p_1-subgroup of H. Let B_2 be a Sylow p_2-subgroup of $P_2 P_1 \cap H$ and note that it is a Sylow p_2-subgroup of H. As $P_1 \cap H$ is normal in H, we have $(P_1 \cap H) B_2 \leq P_2 P_1 \cap H$. By consideration of cardinality $P_2 P_1 \cap H = (P_1 \cap H) B_2$, and so it follows that $(P_1 \cap H) B_2$ is normal in H. Repeating the argument by letting B_3 be a Sylow p_3-subgroup of $(P_3 P_2 P_1 \cap H)$, we have that $(P_1 \cap H) B_2 B_3$ is normal in H, and so on. Thus H possesses the ordered Sylow tower property.

(ii) Let H be a proper normal subgroup of G. Then $P_i H/H$ is a Sylow p_i-subgroup of G/H, $i=1,2, \dots n$. Since $P_k \dots P_2 P_1$ is normal in G for $k = 1, 2, \dots, n$, it follows that $(P_k \dots P_2 P_1) H/H$ is

a normal in G/H. Then $(P_k H/H) \ldots (P_2 H/H)(P_1 H/H)$ is normal in G/H for $k = 1, 2, \ldots, n$. Thus G/H possesses an ordered Sylow tower property.

As an immediate consequence of Theorem 2.3.1 , we have :

Corollary 2.3.2 . If a group G possesses the ordered Sylow tower property , then G is solvable .

Proof . Let G be a group of order $p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$, where $p_1 > p_2 > \ldots > p_n$. Since G possesses the ordered Sylow tower property , we have that P_1 is normal in G, where P_1 is a Sylow p_1-subgroup of G, it follows that G/P_1 possesses an ordered Sylow tower property by Theorem 2.3.1 (ii) . By induction on the order of G, G/P_1 is solvable . Therefore , the solvability of G follows by (2.2.1) .

Theorem 2.3.3 . [**11** , Lemma 4.1] . Let G be a solvable group of order $p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$, where $p_1 > p_2 > \ldots > p_n$. Suppose that H is a normal p_1-subgroup of G such that G/H possesses an ordered Sylow tower property. Then so also does G.

Proof. Let P_i be a Sylow p_i-subgroup of G, where $i = 1, 2, \ldots, n$. Thus for each i, $P_i H/H$ is a Sylow p_i-subgroup of G/H. Since G/H possesses an ordered Sylow tower property , it follows that $(P_k H/H) \ldots (P_2 H/H)(P_1 H/H)$ is normal in G/H for $k = 1, 2, \ldots, n$. Thus $(P_k H) \ldots (P_2 H)P_1 = (P_k \ldots P_2)P_1 H = P_k \ldots P_2 P_1$ is normal in G. Therefore , G possesses an ordered Sylow tower property.

Theorem 2.3.4 . [**11** , Theorem 1.8]. Let G be a supersolvable group of order $p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$. Then G possesses the ordered Sylow tower property.

Proof . We prove the Theorem by induction on the order of G. Let P_1 be a Sylow p_1-subgroup of G. We argue that P_1 is normal in G. Let N be a minimal normal subgroup of G. By induction NP_1/N is a normal subgroup of G/N and hence NP_1 is normal in G. Let $|N| = q$, where q is a prime in $\pi(G)$. If $p_1 = q$, then $NP_1 = P_1$ and we are done; hence we may assume that $p_1 > q$. Since $|N| = q$, it follows by (2.2.2) that $G/C_G(N) \lesssim \mathrm{Aut}(N)$ and by (2.2.3), $|\mathrm{Aut}(N)| = q-1$. Since $p_1 > q$, we have that p_1 does not divide $|\mathrm{Aut}(N)|$ and so $P_1 \leq C_G(N)$. Now it follows that P_1 is a normal subgroup of NP_1 and as P_1 is also a Sylow p_1-subgroup of NP_1, we have that P_1 is characteristic in NP_1. Since P_1 char NP_1 and NP_1 is normal in G, we conclude that P_1 is normal in G. Clearly, G/P_1 possesses an ordered Sylow tower property and so by applying Theorem 2.3.3 , we conclude that G possesses an ordered Sylow tower property.

The following example shows that the converse of Theorem 2.3.4 is not true.

Example 2.3.5 . Let $H = \langle a \rangle \times \langle b \rangle$, with $|a| = |b| = 3$. Then there is an automorphism t of H of order 4 such that $at = b$, $bt = a^{-1}$ and $G = \mathrm{Hol}(H, \langle t \rangle)$ is of order 36 , has a normal Sylow subgroup of order 9 , but no normal subgroup of order 3 . Clearly , G possesses an ordered Sylow tower property but not supersolvable. (see [**10** , Exercise 9.2.13]).

In the proof of the following Theorem, we followed the method of Assad and Shaalan result [3 , Corollary 3.6] .

Theorem 2.3.6 . Suppose that H and K are normal subgroups of G such that $G = HK$ and that H and K possess the ordered Sylow tower property. Then G possesses the ordered Sylow tower property.

Proof . Suppose that the Theorem is false and let G be a counter-example of minimal order. Let p be the largest prime dividing $|G|$. Then there exist a Sylow p-subgroup P_1 of H and a Sylow p-subgroup P_2 of K such that $P = P_1 P_2$ is a Sylow p-subgroup of G by (2.2.4).

Let M be a proper subgroup of G such that $H \leq M$. Since $G = HK$ and $H \leq M$, it follows by (2.2.5) that $M = H(M \cap K)$. By Theorem 2.3.1 (i) , $(M \cap K)$ possesses an ordered Sylow tower property. Now our choice of G implies that M possesses an ordered Sylow tower property. Similarly , if $K \leq M$, then M possesses an ordered Sylow tower property.

Now we consider the following cases :

Case 1 . $P_1 = P$ and $P_2 = 1$. Then P is normal in H as H possesses an ordered Sylow tower property. Thus PK is a subgroup of G. If $PK = G$, then $|G:K| = |P|$. Clearly, $p \mid |G:H|$. Hence $(|G:H|, |G:K|) = 1$. Applying (2.2.6), it follows that G possesses an ordered Sylow tower property ; *a* contradiction. If PK is a proper subgroup of G, then by the second paragraph in this proof, PK possesses an ordered Sylow tower property and so P is normal in PK. Since P is normal in H and P is normal in PK , we have that P is normal in G. By our choice of G , G/P possesses an ordered Sylow tower property. Hence G possesses an ordered Sylow tower property by Theorem 2.3.3 ; a contradiction. Similarly , if $P_2 = P$ and $P_1 = 1$, we have a contradiction.

Case 2 . $P_1 = P$ and $1 < P_2 < P$. Then P_1 is normal in H and P_2 is normal in K. Since $P_2^{kh} = P_2^h \leq P^h = P \leq H$, it follows that $P_2^G \leq' P$. By our choice of G, G/P_2^G possesses an ordered Sylow tower property and so P is normal in G. Hence G possesses an ordered Sylow tower property by Theorem 2.3.3 ; a contradiction. Similarly , if $P_2 = P$ and $1 < P_1 < P$, we have a contradiction.

Case 3 . $1 < P_1 < P$ and $1 < P_2 < P$. Clearly, $P_2 H$ is a subgroup of G. If $G = P_2 H$, then $G = HP$. Hence, if $y \in G$, $y = hk$ with $h \in H$ and $k \in P$, and consequently $P_1^y = P_1^{hk} = P_1^k \leq P_1$. But then the normal closure $P_1^G = \langle P_1^y | y \in G \rangle \leq P$. By our choice of G, G/P_1^G possesses an ordered Sylow tower property and so P is normal in G. Hence G possesses an ordered Sylow tower property by Theorem 2.3.3 ; a contradiction. Similarly, if $G = P_1 K$, we have a contradiction. Thus we can assume that $P_2 H$ and $P_1 K$ are proper subgroups of G. Then by the second paragraph of this proof, $P_2 H$ and $P_1 K$ possess the ordered Sylow tower property. Hence P is normal in $P_2 H$ and P is normal in $P_1 K$ and so P is normal in G. Therefore, G possesses an ordered Sylow tower property ; a final contradiction.

CHAPTER III

ON THE GENERALIZED SYLOW TOWER PROPERTY OF A FINITE GROUP

ON THE GENERALIZED SYLOW TOWER PROPERTY OF A FINITE GROUP

3.1 INTRODUCTION

In this chapter we investigate the conditions under which group possesses a generalized Sylow tower property and we also obtain some properties concerning the generalized Sylow tower property. Furthermore we study the influence of the generalized Sylow tower property of some subgroups of a group on its structure.

3.2 EXISTENCE OF THE GENERALIZED SYLOW TOWER PROPERTY AND SOME FURTHER RESULTS

In this section, we prove the following results :

Theorem 3.2.1 . Let G be a group of order $p_1^{e_1}\ p_2^{e_2}\ \dots\ p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$. Then G possesses the generalized Sylow tower property if and only if for some σ in S_n, $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(G)$ with $i > j$, $1 \leq i, j \leq n$.

Proof . Suppose that G possesses the generalized Sylow tower property . Then for some σ in S_n, we have that $K = P_{\sigma(i-1)} P_{\sigma(i-2)} \dots P_{\sigma(j)} \dots P_{\sigma(1)}$ is normal in G where $P_{\sigma(i)}$ is a Sylow $p_{\sigma(i)}$-subgroup of G . Hence if G contains a $(p_{\sigma(i)}, p_{\sigma(j)})$-subgroup L for some primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(G)$ with $i > j$, then $P_{\sigma(j)} = L \cap K$ is normal in L, where $P_{\sigma(j)}$ is a Sylow $p_{\sigma(j)}$-

subgroup of L ; a contradiction. Thus $(p_{\sigma(i)}, p_{\sigma(j)}) \nleq G$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(G)$ with $i > j$.

Conversely , suppose that for some σ in S_n, $(p_{\sigma(i)}, p_{\sigma(j)}) \nleq G$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(G)$ with $i > j$, $1 \leq i, j \leq n$. Put $i = n$, we have that $(p_{\sigma(n)}, p_{\sigma(j)}) \nleq G$ for $n > j$, $j = n-1,...,1$. Thus by (2.2.7) , we conclude that G has a normal $p_{\sigma(n)}$-complement . Then $G = P_{\sigma(n)} K_n$, where $P_{\sigma(n)}$ is a Sylow $p_{\sigma(n)}$-subgroup of G and K_n is a normal complement of $P_{\sigma(n)}$ in G. By induction on the order of G, K_n possesses the generalized Sylow tower property. Therefore , G possesses the generalized Sylow tower property.

The following results establish the closure property of the class of the generalized Sylow tower property under subgroups , factor groups and the finite direct product.

Theorem 3.2.2 . Let G be a group of order $p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$. Suppose that G possesses the generalized Sylow tower property. Then :

(i) Every subgroup of G possesses the generalized Sylow tower property.

(ii) Every homomorphic image of G possesses the generalized Sylow tower property.

Proof . Since G possesses the generalized Sylow tower property , it follows by Theorem 3.2.1 that for some σ in

S_n, $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(G)$ with $i > j$, $1 \leq i, j \leq n$.

(i) Let H be a subgroup of G. Clearly, we have for the same σ, $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq H$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(H)$ with $i > j$. Hence, Theorem 3.2.1 implies that H possesses the generalized Sylow tower property.

(ii) Let H be a proper normal subgroup of G. Since for the same σ, $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G$ for $i > j$, it follows by (2.2.8) that $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G/H$ for $i > j$. Once again by applying Theorem 3.2.1, we conclude that G/H possesses the generalized Sylow tower property.

As an immediate consequence of Theorem 3.2.2, we have that :

Corollary 3.2.3 . If a group G possesses the generalized Sylow tower property, then G is solvable.

Proof . Let G be a group of order $p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$. Since G possesses the generalized Sylow tower property, then for some σ in S_n, $P_{\sigma(k)} \cdots P_{\sigma(2)} P_{\sigma(1)}$ is normal in G for $k = 1, 2, \dots, n$, where $P_{\sigma(i)}$ is a Sylow $p_{\sigma(i)}$-subgroup of G. In particular, $P_{\sigma(1)}$ is normal in G. Then Theorem 3.2.2 (ii) implies that $G/P_{\sigma(1)}$ possesses the generalized Sylow tower property. By induction on order of G, $G/P_{\sigma(1)}$ is solvable. Thus by (2.2.1), G is solvable.

We prove the following result :

Theorem 3.2.4 . Let H and K be normal subgroups of G such that G/H and G/K possess the generalized Sylow tower property for some σ in S_n. Then $G/H \cap K$ possesses the generalized Sylow tower property for the same σ.

Proof . Let G be a counter-example of minimal order. Then $G/H \cap K$ contains a $(p_{\sigma(i)}, p_{\sigma(j)})$-subgroup $M/H \cap K$ for some primes $p_{\sigma(i)}$, $p_{\sigma(j)}$ in $\pi(G/H \cap K)$ with $i > j$. If $p_{\sigma(i)}$ and $p_{\sigma(j)}$ belong to $\pi(G/H)$ and $\pi(G/K)$, then $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G/H$ and $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G/K$ as G/K and G/H possesses the generalized Sylow tower property, otherwise, we have also $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G/H$ and $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G/K$. Since $MH/H \cong M/H \cap M$ and $MK/K \cong M/K \cap M$, we conclude by Theorem 3.2.2 (i) that $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq M/H \cap M$ and $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq M/K \cap M$. If M is a proper subgroup of G, $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq M/H \cap K$ by our choice of G ; a contradiction. Th[illegible] is a $(p_{\sigma(i)}, p_{\sigma(j)})$-group. Since $G/H \cap K$ [illegible] $, p_{\sigma(j)}) \not\leq G/H$, we conclude that [illegible] se is G/K. Since $G/H \cap K \cong G/H$ [illegible] 2.9 [illegible] follows that $G/H \cap K$ is nilpotent and so $G/$ [illegible] ss [illegible] generalized Sylow tower property ; a final contradiction.

As an immediate [illegible] orem 3.2.4 . We have :

Corollary 3.2 [illegible] , where both H and K possess the generalized Sylow [illegible] for some σ in S_n. Then G possesses the generalized [illegible] to [illegible] operty for the same σ.

Proof . Since $G/H \cong K$ and $G/K \cong H$, it follows that G/H and G/K possess the generalized Sylow tower property for some σ in S_n . Thus by applying Theorem 3.2.4 , we conclude that G possesses the generalized Sylow tower property for the same σ .

We can prove the following result :

Theorem 3.2.6 . Let G be a solvable group of order $p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$, where $p_1 > p_2 > \ldots > p_n$. For some σ in S_n , let H be a normal $p_{\sigma(1)}$-subgroup of G such that G/H possesses the generalized Sylow tower property. Then so also does G .

Proof . Since G/H possesses the generalized Sylow tower property , it follows that $(P_{\sigma(k)} H/H) \ldots (P_{\sigma(2)} H/H)(P_{\sigma(1)} H/H)$ is normal in G/H for $k = 1, 2, \ldots, n$, where $P_{\sigma(i)}$ is a Sylow $p_{\sigma(i)}$-subgroup of G . In particular , $P_{\sigma(1)}/H$ is normal in G/H and so $P_{\sigma(1)}$ is normal in G . Since $(P_{\sigma(k)} H/H) \ldots (P_{\sigma(2)} H/H)(P_{\sigma(1)}/H)$ is normal in G/H , we have that $(P_{\sigma(k)} H) \ldots (P_{\sigma(2)} H)(P_{\sigma(1)})$ is normal in G . Since $(P_{\sigma(k)} H)\ldots(P_{\sigma(2)} H)(P_{\sigma(1)}) = P_{\sigma(k)} \ldots P_{\sigma(2)} P_{\sigma(1)}$, it follows that $P_{\sigma(k)} \ldots P_{\sigma(2)} P_{\sigma(1)}$ is normal in G for $k = 1, 2, \ldots, n$. Therefore , G possesses the generalized Sylow tower property.

It is well known that : A group G is supersolvable if and only if $G/\Phi(G)$ is supersolvable (see[**8** , Satz 8.6(a)]). The following result is analogous to the above result for the generalized Sylow tower property.

Theorem 3.2.7 . Let G be a group of order $p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$. Then G possesses the generalized Sylow tower property if and only if $G/\Phi(G)$ possesses the generalized Sylow tower property.

Proof . It is easy to show that G is a solvable group. Suppose that $G/\Phi(G)$ possesses a generalized Sylow tower property. It follows from applying Theorem 3.2.1 that for some σ in S_n, $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G/\Phi(G)$ for each pair of primes in $\pi(G/\Phi(G))$ with $i > j$. By (2.2.10) , $\pi(G) = \pi(G/\Phi(G))$. Thus by applying (2.2.11), it follows that $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G$ for each pair of primes $p_{\sigma(i)}$, $p_{\sigma(j)}$ in $\pi(G)$ with $i > j$. Therefore, G possesses the generalized Sylow tower property by Theorem 3.2.1 .

Conversely , suppose that G possesses the generalized Sylow tower property. Theorem 3.2.2 (ii) implies that $G/\Phi(G)$ possesses the generalized Sylow tower property.

Remark .2 . Let C be a class of all groups which have the generalized Sylow tower property. This is a formation by Theorem3.2.2 (ii) and Theorem 3.2.4 . Moreover , Theorem 3.2.7 shows that C is saturated.

Now we can prove the following result :

Theorem 3.2.8 . Let H be a π-Hall subgroup of a solvable group G such that $\Phi(G) \leq H$. Then H possesses the generalized Sylow tower property if and only if $H/\Phi(G)$ possesses the generalized Sylow tower property.

Proof. Suppose that $H/\Phi(G)$ possesses the generalized Sylow tower property. It follows from applying Theorem 3.2.1 that for some σ in S_n, $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq H/\Phi(G)$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(H/\Phi(G))$ with $i > j$. By (2.2.10), $\pi(G) = \pi(G/\Phi(G))$ and since H is a π-Hall subgroup of G, it follows that $\pi(H) = \pi(H/\Phi(G))$. Hence $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq H/\Phi(G)$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(H)$ with $i > j$. We argue that $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G/\Phi(G)$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(H/\Phi(G))$ with $i > j$. If not, there exists a $(p_{\sigma(i)}, p_{\sigma(j)})$-subgroup $K/\Phi(G)$ in $G/\Phi(G)$ for some primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(H/\Phi(G))$ with $i > j$. By (2.2.12) there exists a Hall subgroup $L/\Phi(G)$ such that $K/\Phi(G) \leq L/\Phi(G)$, where $\pi(L/\Phi(G)) = \{p_{\sigma(i)}, p_{\sigma(j)}\}$. By (2.2.12), we may assume that $L/\Phi(G) \leq H/\Phi(G)$. This is a contradiction as $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq H/\Phi(G)$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(H/\Phi(G))$ with $i > j$. Thus $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G/\Phi(G)$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(H/\Phi(G))$ with $i > j$. By (2.2.11), $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(H/\Phi(G))$ with $i > j$ and since $\pi(H/\Phi(G)) = \pi(H)$, it follows that $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq G$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(H)$ with $i > j$. Hence $(p_{\sigma(i)}, p_{\sigma(j)}) \not\leq H$ for each pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(H)$ with $i > j$. Now, Theorem 3.2.1 implies that H possesses the generalized Sylow tower property.

Conversely, suppose that H possesses the generalized Sylow tower property. Applying Theorem 3.2.2 (ii), $H/\Phi(G)$ possesses the generalized Sylow tower property.

3.3 THE INFLUENCE OF THE GENERALIZED SYLOW TOWER PROPERTY OF SOME SUBGROUPS ON THE STRUCTURE OF A FINITE GROUP

In this section, we prove the following results :

Theorem 3.3.1 . Let G be a group of order $p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$, where $p_1 > p_2 > \ldots > p_n$. Suppose that G has three subgroups H_1, H_2 and H_3 whose indices in G are pairwise relatively prime. If H_1, H_2 and H_3 possess the generalized Sylow tower property for some σ in S_n, then G possesses the generalized Sylow tower property for the same σ.

Proof. We prove the Theorem by induction on the order of G. If $H_1 = 1$, then $|G:H_1| = |G|$. Then $(|G:H_2|,|G|) = 1$ which implies that $G = H_2$, whence G possesses the generalized Sylow tower property by hypothesis. Thus we may assume that $H_i \neq 1$, $i = 1,2,3$. Since H_1, H_2 and H_3 possess the generalized Sylow tower property, it follows by Corollary 3.2.3 that H_1, H_2 and H_3 are solvable groups. Then G is solvable by (2.2.13). Let M be an arbitrary proper maximal subgroup of G. We argue that M possesses the generalized Sylow tower property. Since G is solvable, we have by (2.2.14) that $|G:M| = p^n$ for some prime p.

Now we deal with the following two cases :

Case1. $(|G:M|,|G:H_i|) = 1$ for each $i = 1,2,3$. Then $G = MH_i$, $i = 1,2,3$ by (2.2.15) and so $|G:H_i| = |M:M \cap H_i|$. Since $M \cap H_i$ possesses the generalized Sylow tower property by Theorem 3.2.2 (i), it

follows that M possesses the generalized Sylow tower property by induction on the order of G.

Case 2. $(|G:M|,|G:H_i|) \neq 1$ for some i. Since $|G:M|$ and $|G:H_i|$ are not relatively prime for some i, we can assume that $H_i = H_1$. Since $|G:M| = p^n$, then $p \mid |G:H_1|$ and so $p \nmid |H_1|$ or $p \mid |H_1|$. If $p \nmid |H_1|$, then we may assume that $H_1 \leq M$ by (2.2.12). Clearly, $(|M:H_1|,|M:M \cap H_i|) = 1$ for $i = 2, 3$. Hence M possesses the generalized Sylow tower property under the assumption that $p \nmid |H_1|$. Thus we may assume that $p \mid |H_1|$. Then by (2.2.16), $H_1 = PL$, where P is a Sylow p-subgroup of H_1 and L is a p`-Hall subgroup of H_1. Since $L < H_1 < G$, we have :

$$|G:L| = |G:H_1| \; |H_1:L| \qquad (1)$$

As G is solvable, so G has a p-complement N by (2.2.16), we may assume that $L \leq N \leq M$ by (2.2.12). It is clear that :

$$|G:L| = |G:M| \; |M:L| \qquad (2)$$

From (1) and (2) it follows that $\pi(|G:H_1|) \geq \pi(|M:L|)$, and hence $(|M:L|,|M:M \cap H_i|) = 1$ for $i = 2, 3$. Therefore M possesses the generalized Sylow tower property by induction on the order of G. Since M is an arbitrary maximal subgroup of G and M possesses the generalized Sylow tower property, we have that all proper subgroups of G possess the generalized Sylow tower property. If G does not possess the generalized Sylow tower property, then for some σ in S_n, G contains an $(p_{\sigma(i)}, p_{\sigma(j)})$-subgroup N for some pair of primes $p_{\sigma(i)}$ and $p_{\sigma(j)}$ in $\pi(G)$ with $i > j$. Thus by Theorem 3.2.1, N does not possesses the generalized Sylow tower property; a contradiction. Therefore, G possesses the generalized Sylow tower property.

As an immediate consequence of Theorem 3.3.1 , we have :

Corollary 3.3.2 . [**2** , Theorem 11] . Let G be a finite group. Suppose that G has three subgroups H_1, H_2 and H_3 whose indices in G are pairwise relatively prime . If H_1 , H_2 and H_3 possess the ordered Sylow tower property, then G possesses an ordered Sylow tower property.

The following result gives the structure of minimal non generalized Sylow tower property (A group does not possess the generalized Sylow tower property but whose all proper subgroups possess the generalized Sylow tower property) .

Theorem 3.3.3 . If G does not possess the generalized Sylow tower property and all proper subgroups of G possess the generalized Sylow tower property , then there exist a Sylow p-subgroup P of G and a Sylow q-subgroup Q of G for some distinct primes p and q such that **(i)** $G = PQ$, **(ii)** P is normal in G . **(iii)** Q is a non-normal cyclic.

Proof. We argue that $|\pi(G)| = 2$. If not , then $|\pi(G)| \geq 3$. Then Theorem 3.3.1 implies that G possesses the generalized Sylow tower property; a contradiction. Thus $|\pi(G)| = 2$. So $G = PQ$, where P is a Sylow p-subgroup of G and Q is a Sylow q-subgroup of G for distinct primes p and q . Clearly , one of them is normal and the other is non-normal. Thus G is an (p,q)-group or (q,p)-group. Hence G is non-nilpotent and all proper subgroups are nilpotent. Thus by (2.2.17) , P is normal in G and Q is a non-normal cyclic..

As an immediate consequence of Theorem 3.3.3 , we have :

Corollary 3.3.4 . A group G is a minimal non-generalized Sylow tower if and only if G is a minimal non-nilpotent group.

Proof . Suppose that G is a minimal non-generalized Sylow tower property. Theorem 3.3.3 implies that $G = PQ$, where P is a Sylow p-subgroup of G and Q is a non-normal Sylow q-subgroup of $G, p \neq q$. Then G is (p,q)-group. Thus G is a non-nilpotent and all proper subgroups are nilpotent, that is, G is a minimal non-nilpotent group.

Now we may assume that G is a minimal non-nilpotent group. Then G does not possess the generalized Sylow tower property and all proper subgroups possess the generalized Sylow tower property, that is, G is a minimal non-generalized Sylow tower property.

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ARABIC SUMMARY

Kingdom of Saudi Arabia
Ministry of Higher Education
Umm Al – Qura University
Faculty of Applied Sciences
Department of Mathematical
Sciences

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By
WAFA MOHAMED FAGIEH

Supervised by
Dr. MOHAMED RAMADAN SHAHEEN
Umm Al - Qura University
Faculty of Applied Science
Department of Mathematics

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